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Representations of Finite Groups of Lie Type

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INTRODUCTION

These notes follow a course given at the University of Paris VII during the spring semester of academic year 1987–88. Their purpose is to expound basic results in the representation theory of finite groups of Lie type (a precise definition of this concept will be given in chapter 3).

Let us start with some notations. We denote by $\mathbf{F}_q$ a finite field of characteristic p with q elements (q is a power of p). The typical groups we will look at are the linear, unitary, symplectic, orthogonal, ... groups over $\mathbf{F}_q$. We will consider these groups as the subgroups of points with coefficients in $\mathbf{F}_q$ of the corresponding groups over the algebraic closure $\overline{\mathbf{F}}_q$ (which are algebraic reductive groups). More precisely, the group over $\mathbf{F}_q$ is the set of fixed points of the group over $\overline{\mathbf{F}}_q$ under an endomorphism F called the *Frobenius endomorphism*; this will be explained in chapter 3. In the following paragraphs of this introduction we will try to describe, by some examples, a sample of the methods used to study the complex representations of these groups. More examples are developed in detail in chapter 15.

Induction from subgroups

Let us start with the example where $G = \mathbf{GL}_n(\mathbf{F}_q)$ is the general linear group over $\mathbf{F}_q$. Let T be the subgroup of diagonal matrices; it is a subgroup of the group B of upper triangular matrices, and there is a semi-direct product decomposition $B = U \rtimes T$, where U is the subgroup of the upper triangular matrices which have all their diagonal coefficients equal to 1. The representation theory of T is easy since it is a commutative group (actually isomorphic to a product of n copies of the multiplicative group $\mathbf{F}_q^\times$). Composition with the natural homomorphism from B to T (quotient by U) lifts representations of T to representations of B. Inducing these representations from B to the whole of the linear groups gives representations of G (whose irreducible components are called "principal series representations"). More generally we can replace T with a group L of block-diagonal matrices, B with the group of corresponding upper block-triangular matrices P, and we have a semi-direct product decomposition (called a Levi decomposition) $P = V \rtimes L$, where V is the subgroup of P whose diagonal blocks are identity matrices; we may as before induce from P to G representations of L lifted to P. The point of this method is that L is isomorphic to a direct product of linear groups of smaller degrees than n. We thus have an inductive process to get representations of G if we know how to decompose induced representations from P to G. This approach has been developed in the works of Harish-Chandra, Howlett and Lehrer, and is introduced in chapters 4 to 7.

Cohomological methods

Let us now consider the example of $G = U_n$, the unitary group over $\mathbf{F}_q$. It can be defined as the subgroup of matrices $A \in \mathbf{GL}_n(\mathbf{F}_{q^2})$ such that ${}^tA^{[q]} = A^{-1}$, where $A^{[q]}$ denotes the matrix whose coefficients are those of A raised to the q-th power. It is thus the subgroup of $\mathbf{GL}_n(\overline{\mathbf{F}}_q)$ consisting of the fixed points of the endomorphism $F : A \mapsto ({}^tA^{[q]})^{-1}$.

A subgroup L of block-diagonal matrices in U_n is again a product of unitary groups of smaller degree. But this time we cannot construct a bigger group P having L as a quotient. More precisely, the group $\mathbf{V}$ of upper block-triangular matrices with coefficients in $\overline{\mathbf{F}}_q$ and whose diagonal blocks are the identity matrix has no fixed points other than the identity under F.

To get a suitable theory, Harish-Chandra's construction must be generalized; instead of inducing from $V \rtimes L$ to G, we construct a variety attached to $\mathbf{V}$ on which both L and G act with commuting actions, and the cohomology of that variety with ℓ-adic coefficients gives a (virtual) bi-module which defines a "generalized induction" from L to G. This approach, due to Deligne and Lusztig, will be developed in chapters 10 to 13.

Gelfand-Graev representations

Using the above methods, a lot of information can be obtained about the characters of the groups $\mathbf{G}(\mathbf{F}_q)$, when $\mathbf{G}$ has a connected centre. The situation is not so clear when the centre of $\mathbf{G}$ is not connected. In this case one can use the Gelfand-Graev representations, which are obtained by inducing a linear character "in general position" of a maximal unipotent subgroup (in $\mathbf{GL}_n$ the subgroup of upper triangular matrices with ones on the diagonal is such a subgroup). These representations are closely tied to the theory of regular unipotent elements. They are multiplicity-free and contain rather large cross-sections of the set of irreducible characters, so give useful additional information in the non-connected centre case (in the connected centre case, they are combinations of Deligne-Lusztig characters).

For instance, in $\mathbf{SL}_2(\mathbf{F}_q)$ they are obtained by inducing a non-trivial linear character of the group of matrices of the form $\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}$: such a character corresponds to a non-trivial additive character of $\mathbf{F}_q$; there are two classes of such characters under $\mathbf{SL}_2(\mathbf{F}_q)$, which corresponds to the fact that the centre of $\mathbf{SL}_2$ has two connected components (its two elements).

The theory of regular elements and Gelfand-Graev representations is expounded in chapter 14, with, as an application, the computation of all irreducible characters on regular unipotent elements.

Assumed background

We will assume that the reader has some basic familiarity with algebraic geometry, but we will give as far as possible statements of all the results we use; a possible source for these is R. Hartshorne's book "Algebraic Geometry" ([Ha]). Chapters 0, 1 and 2 contain the main results we use from the theory of algebraic groups (the proofs will be often omitted in chapter 0; references for them may be found in the books on algebraic groups by A. Borel [B1], J. E. Humphreys [Hu] and T. A Springer [Sp] which are all good introductions to the subject). We will also recall results about Coxeter groups and root systems for which the most convenient reference is the volume of N. Bourbaki [Bbk] containing chapters IV, V and VI of the theory of Lie groups and Lie algebras. However, we will not give any references for the basic results of the theory of representations of finite groups over fields of characteristic 0, which we assume known. All we need is covered in the first two parts of the book of J. -P. Serre [Se].

Bibliography

Appropriate references will be given for each statement. There are two works about the subject of this book that we will not refer to systematically, but which the reader should consult to get additional material: the book of B. Srinivasan [Sr] for the methods of Deligne and Lusztig, and the survey of R. W. Carter [Ca] which covers many topics that we could not introduce in the span of a one-semester course (such as unipotent classes, Hecke algebras, the work of Kazhdan and Lusztig, ...); furthermore our viewpoint or the organization of our proofs are often quite different from Carter's (for instance the systematic use we make of Mackey's formula (chapters 5 and 11) and of Curtis-Kawanaka-Lusztig duality). To get further references, the reader may look at the quasi-exhaustive bibliography on the subject up to 1986 which is in Carter's book.

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